Big problems for small heights: an introduction to Mahler's measure

Riccardo Pengo

Stellenbosch University, 28-30 March 2022

1 What

This mini-course aims at providing an introduction to the world of Mahler's measure, motivated by Lehmer's problem. We will start by looking at the univariate case, and then move to the multivariate setting, focusing in the last part on the fascinating relations between Mahler measures and special values of L-functions.

Let us be more precise. Formulated almost a century ago in the seminal paper [13], Lehmer's question still remains one of the most challenging unsolved tasks in number theory. The main protagonist featured in this problem is *Mahler's measure*, an invariant initially studied by Pierce [17] and Lehmer [13] for univariate polynomials, whose definition was later generalised by Mahler [14] to the multivariate setting.

Despite the extensive efforts of many researchers, Lehmer's problem remains open to this day, even if some steps towards its solutions have been made by Dobrowolski [10] and Smyth [23]. Much more recently, Dimitrov [9] provided an ingenious solution to a related problem, formulated by Schinzel and Zassenhaus [22].

In a slightly different direction, Boyd [2] proposed a strategy to solve Lehmer's question based on the limit relations between the Mahler measures of univariate and multivariate polynomials. These relations were later made more precise in the work of Lawton [12], which we generalised to sequences of multivariate polynomials in a recent work written jointly with Brunault, Guilloux and Mehrabdollahei.

The peculiar role played by multivariate Mahler measures in Boyd's approach towards Lehmer's problem prompted Smyth [24] to compute the explicit values for the Mahler measures of some simple linear forms in two and three variables. Perhaps surprisingly, he found out that these Mahler measures were related to the values at $s = 2$ of a Dirichlet L-function, and at $s = 3$ of Riemann's ζ-function. The study of Mahler measures of linear forms was further continued by Cassaigne and Maillot [15, § 7.3], who connected them to the Bloch-Wigner dilogarithm [26], and thus to the world of hyperbolic geometry. More recently, Rodriguez-Villegas, Toledano and Vaaler [19] provided explicit formulas for these linear Mahler measures, and Borwein, Straub, Wan and Zudilin [1] linked linear Mahler measures to the densities of uniform random walks in the plane.

Going back to the 1980s, the remarkable identities proved by Smyth prompted Boyd [3] to carry out extensive numerical computations on Mahler measures for families of polynomials in two variables, looking for identities between them and more complicated L-functions. Some of these identities, relating Mahler measures of bivariate polynomials to the special value at $s = 2$ of L-functions associated to elliptic curves, were linked to modular forms in the work of Rodriguez-Villegas [18], and proved much more recently using a clever idea of Rogers and Zudilin [20, 21], later generalized in the work of Brunault [5, 6] and in the PhD thesis of Wang [25].

Finally, a fascinating link between the numerical identities provided by Boyd's work and the deep conjectures of Beilinson concerning special values of L-functions was found by Deninger [8]. His approach, however, did not manage to give a conceptual explanation for the identities proved by Smyth, which started the whole story! Some ideas towards such a conceptual explanation were put forward by Maillot [4, § 8] and Lalín [11], and are the subject of recent joint work with Brunault (see [16, Chapter 5] for a summary).

2 When

The mini-course will consist of six lectures of one hour each, and will take place between the 28th and the 30th of March, 2022.

3 Who

The lectures will be accessible to master and doctoral students with an interest in number theory and algebraic geometry.

4 Where

The lectures will take place in the Mathematics Department of Stellenbosch University (South Africa).

5 How

The main reference for this mini-course will be the recent book [7] by Brunault and Zudilin, and occasional references may be made to the works mentioned in Section 1.

The following provides a tentative schedule for the topics covered in each lecture, which may be changed according to the interests of the participants:

1. Mahler measures of univariate polynomials, Jensen's formula and Lehmer's problem;
2. Partial results towards Lehmer's problem: the theorems of Smyth, Dobrowolski and Dimitrov;
3. Lehmer's problem and limits of multivariate Mahler measures;
4. Mahler measures of linear forms: explicit formulas, asymptotics and conjectures;
5. Mahler measures and elliptic curves: an introduction to the Rogers-Zudilin method;
6. Mahler measures as periods: Beilinson's conjectures and exact polynomials.

Given the amount of material to be covered, the lectures will mainly strive to convey the key points of the proofs involved, while skipping some technical parts.

References

[1] J. M. Borwein, A. Straub, J. Wan, and W. Zudilin. "Densities of Short Uniform Random Walks". In: *Canadian Journal of Mathematics* 64.5 (Oct. 2012). With an appendix by D. Zagier, pp. 961–990 (cit. on p. 1).

[2] D. W. Boyd. "Speculations concerning the range of Mahler's measure". In: *Canadian Mathematical Bulletin* 24.4 (1981), pp. 453–469 (cit. on p. 1).

[3] D. W. Boyd. "Mahler's measure and special values of L-functions". In: *Experimental Mathematics* 7.1 (1998), pp. 37–82 (cit. on p. 1).

[4] D. W. Boyd, D. Lind, F. Rodriguez Villegas, and C. Deninger. "The many aspects of Mahler's measure". Final report of the Workshop 03w5035 at the Banff International Research Station, Banff, Canada. (May 2003) (cit. on p. 1).

[5] F. Brunault. "Regulators of Siegel units and applications". In: *Journal of Number Theory* 163.Supplement C (June 2016), pp. 542–569 (cit. on p. 1).

[6] F. Brunault. "Régulateurs modulaires explicites via la méthode de Rogers–Zudilin". In: *Compositio Mathematica* 153.6 (June 2017), pp. 1119–1152 (cit. on p. 1).

[7] F. Brunault and W. Zudilin. *Many Variations of Mahler Measures: A Lasting Symphony*. Australian Mathematical Society Lecture Series. Cambridge University Press, 2020 (cit. on p. 2).

[8] C. Deninger. "Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions". In: *Journal of the American Mathematical Society* 10.2 (1997), pp. 259–281 (cit. on p. 1).

[9] V. Dimitrov. "A proof of the Schinzel-Zassenhaus conjecture on polynomials". In: *arXiv e-Prints* (Dec. 2019) (cit. on p. 1).

[10] E. Dobrowolski. "On a question of Lehmer and the number of irreducible factors of a polynomial". In: *Acta Arithmetica* 34 (1979), pp. 391–401 (cit. on p. 1).

[11] M. N. Lalín. "An algebraic integration for Mahler measure". In: *Duke Mathematical Journal* 138.3 (June 2007). Zbl: 1196.11096, pp. 391–422 (cit. on p. 1).

[12] W. M. Lawton. "A problem of Boyd concerning geometric means of polynomials". In: *Journal of Number Theory* 16.3 (June 1983), pp. 356–362 (cit. on p. 1).

[13] D. H. Lehmer. "Factorization of certain cyclotomic functions". In: *Annals of Mathematics. Second Series* 34.3 (1933), pp. 461–479 (cit. on p. 1).

[14] K. Mahler. "On Some Inequalities for Polynomials in Several Variables". In: *Journal of the London Mathematical Society* s1-37.1 (1962), pp. 341–344 (cit. on p. 1).

[15] V. Maillot. "Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables". In: *Mémoires de la Société Mathématique de France. Nouvelle Série* 80 (2000), pp. vi+129 (cit. on p. 1).

[16] R. Pengo. "Mahler measures, special values of L-functions and complex multiplication". PhD thesis. University of Copenhagen, Oct. 2020 (cit. on p. 1).

[17] T. A. Pierce. "The numerical factors of the arithmetic forms $\prod_i = 1^n(1 \pm \alpha_i^m)$". In: *Annals of Mathematics. Second Series* 18.2 (1916), pp. 53–64 (cit. on p. 1).

[18] F. Rodriguez-Villegas. "Modular Mahler Measures I". In: *Topics in Number Theory: In Honor of B. Gordon and S. Chowla*. Ed. by S. D. Ahlgren, G. E. Andrews, and K. Ono. Mathematics and Its Applications. Springer US, 1999, pp. 17–48 (cit. on p. 1).

[19] F. Rodriguez-Villegas, R. Toledano, and J. D. Vaaler. "Estimates for Mahler's measure of a linear form". In: *Proceedings of the Edinburgh Mathematical Society* 47.2 (June 2004), pp. 473–494 (cit. on p. 1).

[20] M. Rogers and W. Zudilin. "From L-series of elliptic curves to Mahler measures". In: *Compositio Mathematica* 148.2 (Mar. 2012), pp. 385–414 (cit. on p. 1).

[21] M. Rogers and W. Zudilin. "On the Mahler Measure of 1+X+1/X+Y+1/Y". In: *International Mathematics Research Notices* 2014.9 (Jan. 2014), pp. 2305–2326 (cit. on p. 1).

[22] A. Schinzel and H. Zassenhaus. "A refinement of two theorems of Kronecker." In: *The Michigan Mathematical Journal* 12.1 (Apr. 1965). Zbl: 0128.03402, pp. 81–85 (cit. on p. 1).

[23] C. J. Smyth. "On the Product of the Conjugates outside the unit circle of an Algebraic Integer". In: *Bulletin of the London Mathematical Society* 3.2 (July 1971), pp. 169–175 (cit. on p. 1).

[24] C. J. Smyth. "On measures of polynomials in several variables". In: *Bulletin of the Australian Mathematical Society* 23.1 (Feb. 1981), pp. 49–63 (cit. on p. 1).

[25] W. Wang. "Regularized integrals and L-functions of modular forms via the Rogers-Zudilin method". PhD thesis. École normale supérieure de Lyon, Sept. 2020 (cit. on p. 1).

[26] D. Zagier. "The Dilogarithm Function". In: *Frontiers in Number Theory, Physics, and Geometry II*. Springer, Berlin, Heidelberg, 2007, pp. 3–65 (cit. on p. 1).